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Optimal collective dichotomous choice under partial order constraints

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Abstract

This paper generalizes optimal collective dichotomous choices by including constraints which limit combinations of acceptance and rejection decisions for *m* projects under consideration. Two types of constraints are examined. The first type occurs when acceptance of some projects requires acceptance of others. This type reduces the choice problem to the tractable (solvable in polynomial time) problem of finding a maximum weight closed subset of a directed acyclic graph. The second type occurs when some projects must be accepted when certain others are rejected. We show that this type renders the choice problem to be NP-complete by reduction from the problem of Vertex Cover. Applicability of the generalization to information filtering is discussed.

Keywords: Constraints; Optimal collective dichotomous choice; Information filtering

JEL classification: D70; D81

1. Introduction

This paper focuses on committees that face dichotomous choices, such as accepting or rejecting investment projects. Optimal group decision-making in a committee of fixed size, that faces uncertain dichotomous choices, has been extensively studied in Grofman

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et al. (1983), Klevorick et al. (1984), Nitzan and Paroush (1982, 1984a,b), Sah (1990, 1991), Sah and Stiglitz (1985, 1986, 1988a,b), Shapley and Grofman (1984), Pete et al. (1993), and Young (1995), as well as many others. Recently, Ben-Yashar and Nitzan (1997) presented a general dichotomous choice framework and derived the optimal decision rule. However, these papers did not study the issue of decision-making under constraints, which is more relevant in economic settings.

The current paper presents optimal collective dichotomous choices under partial order constraints in the general dichotomous choice framework. Choice under partial order constraints is a problem which is often encountered in economic settings, but which can be found in other areas as well. Although we focus on committee decisions, our results are applicable also to problems in which projects with a given value are being considered, such as cost–benefit analysis. We focus first on partial order constraints when the approval of some projects must precede the decisions to approve others (the first type of partial order). For example, the decision to build an overpass must be preceded by a decision to widen the highway. Similarly, constructing holiday resorts in exotic places would be impossible unless convenient access roads for tourists and heavy equipment were constructed initially. We propose a reduction of this problem to a known problem of finding a maximum weight closed subset (Ahuja et al., 1993). The solution to the latter is efficient and tractable (polynomial in the number of projects). We consider also partial order constraints when some projects must be approved if others have been rejected (the second type of partial order). For example, if a decision has been made not to build a by-pass, then lights must be erected. Also, if a hotel decides not to provide access to the sea front, then it must build a swimming pool. We prove that this problem is NP-complete (i.e. which is, in general, considered to be intractable). It is clear that the problem which constitutes a combination of the two types of partial order constraints, will also be NP-complete. A similar, but somewhat different, problem is presented in Page (1997). Where we are concerned with partial order constraints, Page dealt with positive complementarities that exist among projects. He has suggested an appending algorithm which is, in general, intractable. Furthermore, we present an optimal way for aggregating the various decisions made by the individuals, while Page relates to a given value of each project.[1]

In addition, we expand the application of the collective dichotomous model by developing an approach to information filtering problems.

The outline of the paper is as follows: in the next section, the general dichotomous choice model is introduced. In Section 3 we present the problem. In Section 4 we present a solution for an optimal collective decision-making process, while preserving the first type of a given partial order, and we prove that the problem associated with the second type of partial order is NP-complete. Application of the model of information filtering is presented in Section 5. The last section contains a brief summary.

[1]Our problem does not belong to the class of functions UNI and POS discussed in Page (1997), nor does Page assume that the utility from a set of projects is the sum of the utilities of the different projects, as we do.

2. The model

There are m projects (tasks), $N = \{t_1, t_2, \ldots, t_m\}$, and the committee must decide for every project whether to accept or reject it. There are n members in a committee, whose task is to accept or reject projects (tasks). The decision-makers share a common objective — maximizing the committee's expected profit. The collective decision is based on the decisions of the individual members. There are two types of projects, good ones (1) and bad ones (-1). Let s_i denote the state of a project t_i. $s_i = 1$ and $s_i = -1$ are referred to as the two possible states of nature with respect to project t_i, indicating that t_i is a good or bad project, respectively. Therefore, for each project there are two possibilities for making a correct collective decision, namely: (1/1) — accept a good project, and ($-1/-1$) — reject a bad project. The two possibilities of making an incorrect collective decision are ($1/-1$) — accept a bad project, and ($-1/1$) — reject a good project.

The profit associated with the acceptance (1) of a good project t_i is denoted by $B_i(1{:}1)$. The profit associated with the rejection (-1) of a good project t_i is denoted by $B_i(-1{:}1)$, where $B_i(1{:}1) > B_i(-1{:}1)$. Similarly, the profits of the two remaining possibilities for project t_i are denoted by $B_i(-1{:}-1)$ and $B_i(1{:}-1)$, where $B_i(-1{:}-1) > B_i(1{:}-1)$. Note that these profits can differ for different projects. Let $B_i(1) = B_i(1{:}1) - B_i(-1{:}1)$ be the positive net profit when $s_i = 1$. $B_i(-1) = B_i(-1{:}-1) - B_i(1{:}-1)$ is the positive net profit when $s_i = -1$. Let α_i be the a priori probability that project t_i is a good one, $0 < \alpha_i < 1$. Again, this probability can differ for different projects. Let us denote by x_j^i, $x_j^i \in \{1, -1\}$, the decision of committee member j concerning project t_i. $x_j^i = 1$ and $x_j^i = -1$ denote, respectively, acceptance and rejection of project t_i by committee member j. The vector $\bar{x}^i = (x_1^i, \ldots, x_n^i)$ is referred to as the decision profile of the committee members for project t_i. The decisional skill of member j concerning project t_i is characterized by the probabilities p_{ji}^1 and p_{ji}^2 that he accepts a good project and rejects a bad project, respectively. That is, $p_{ji}^1 = \Pr(1{:}1)_{ji}$ and $p_{ji}^2 = \Pr(-1{:}-1)_{ji}$. The probabilities $(1 - p_{ji}^1)$ and $(1 - p_{ji}^2)$ can be interpreted as Type I and Type II errors entailed in individual j's decision regarding project t_i. We assume that each individual has some, but not perfect, decisional skills, $0 < p_{ji}^1 < 1$, $0 < p_{ji}^2 < 1$, $p_{ji}^1 > (1 - p_{ji}^2)$, and that decisions are independent across individuals.[2] Note that the assumption that decisions across individuals are independent is plausible and rational, because the paper discusses the optimal collective rule (process) and as shown by Austen-Smith and Banks (1996) and by Ben-Yashar (2000), independent decisions constitute a Nash equilibrium if and only if the optimal collective rule is used.[3]

[2] $p_{ji}^1 > (1 - p_{ji}^2)$ means that for any project t_i a committee member j is more likely to accept a good project than a bad project, i.e. that the simple average of his decisional skills in the two states of nature exceeds 1/2.

[3] Recently, several authors (e.g. Feddersen and Pesendorfer (1996, 1997, 1998), McLenan (1998), Myerson (1998) and Wit (1998)) have examined the choice model under rational behavior, relaxing the assumption that decision makers vote non-strategically. However, as mentioned above, these discussions are not relevant when dealing with the optimal collective rule.

The committee's decision regarding project t_i is a function of the profile of decisions $\bar{x}^i$ that is associated with this project. Optimization of the committee's collective decision-making process with respect to each project and, in particular, the selection of an aggregation rule (a function that assigns 1 or -1, acceptance or rejection of a project t_i, to any profile of decisions $\bar{x}^i$ in $\Omega = \{1, -1\}^n$) that maximizes expected payoffs has been studied in Ben-Yashar and Nitzan (1997).[4] Our focus is on the optimization of the committee's collective decision-making process when there are constraints, such as preserving a given partial order between the projects when a project may be accepted only if all its predecessors have been accepted or when a project must be accepted if other projects were rejected.

3. The problem

The committee needs to choose projects (tasks) so that the expected profit is maximized, while preserving a given partial order between the projects. That is, the committee must optimally decide which projects to accept. However, in the first type of partial order, the decision to accept a project requires the acceptance of all the projects preceding it. For example, the decision to build an overpass must be preceded by a decision to widen the highway.

In general, there may be situations in which t_j cannot be accepted unless t_i has been accepted. Such a restriction may be represented by a partial order between the projects,[5] where $t_i \preceq t_j$ means that t_i is a predecessor of t_j, and t_j is a successor of t_i. Note that this type of partial order constraints can be considered as 'OR' constraints. This can be shown in the following way, $t_i \preceq t_j$ implies that if t_i is accepted then t_j can be approved, that is, $t_j \rightarrow t_i$ and this is equivalent to, $\neg t_j \vee t_i$.

In the second type of partial order, a project must be accepted if other projects were rejected. For example, if a decision has been made not to build a by-pass, then lights must be erected. Also, if a hotel decides not to provide access to the sea front it must build a swimming pool.

In general, there may be situations in which t_i must be accepted if t_j has been rejected. Such a restriction may be represented by a partial order between the projects, where $t_j \preceq \neg t_i$ means that t_j is the predecessor of $\neg t_i$, in other words, $\neg t_j$ requires the acceptance of t_i. Note that this type of partial order constraints can be also considered as 'positive OR' constraints. This can be shown to be equivalent to $t_j \vee t_i$ implying that the final decision must approve either t_i or t_j (or both).

The committee's decision regarding a project is made by means of an aggregative decision rule. A decision aggregation rule in this case is a function f that assigns a decision profile of 1 and -1, while preserving the given order constraints. In the first

[4]Note that the present discussion pertains to the most general dichotomous model (Ben-Yashar and Nitzan, 1997). A similar discussion could have been developed for the reduced models that have been presented in the past, such as the assumption of a symmetric model (Nitzan and Paroush, 1982; Shapley and Grofman, 1984) or the assumption of homogeneous individuals (Sah, 1990, 1991; Sah and Stiglitz, 1988a,b).

[5]A partial order is a reflexive, anti-symmetric, and transitive order.

type of partial order, f assigns either a decision of 1 or -1 for each project, such that if $f_i = 1$, then there is no $t_j \preceq t_i$ such that $f_j = -1$, where for any project ℓ, $1 \leq \ell \leq m$, f_ℓ is the collective decision for project t_ℓ. That is, $t_i \preceq t_j \Rightarrow f_i \geq f_j$. In the second type of partial order, f must assign a decision of 1 for a project t_i if there is a t_j such that $f_j = -1$ and $t_j \preceq \neg t_i$. The rule f is a function from the set $(\{-1,1\}^n)^m$ of all combinations of m decision profiles, each profile relating to a separate project.[6]

To define formally the function f, we need to present for each project the conditional probabilities of collectively accepting a good project or rejecting a bad one. For each project t_i, we divide the set of all combinations of the decision profiles into two sets $X_i(1/f)$ and $X_i(-1/f)$, for which the decision rule assigns 1 and -1, respectively, to project t_i. A combination of decision profiles is given by $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots, \bar{x}^m)$, where $\bar{x}^i = (x_1^i, x_2^i, \ldots, x_n^i)$, such that

$$X_i(1/f) = \{(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) | f_i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) = 1\} \tag{1}$$

and

$$X_i(-1/f) = \{(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) | f_i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) = -1\}, \tag{2}$$

where f_i is the collective decision for project t_i.

We denote the correct collective probability derived from the decision makers' decisions about project t_i, by $\Pi_i(f:1)$ and by $\Pi_i(f:-1)$, in the two states of nature, $s_i = 1$ and $s_i = -1$

$$\Pi_i(f:1) = \Pr\{(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) \in X_i(1/f) | s_i = 1\} \tag{3}$$

$$\Pi_i(f:-1) = \Pr\{(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) \in X_i(-1/f) | s_i = -1\} \tag{4}$$

and

$$1 - \Pi_i(f:1) = \Pr\{(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) \in X_i(-1/f) | s_i = 1\} \tag{5}$$

$$1 - \Pi_i(f:-1) = \Pr\{(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) \in X_i(1/f) | s_i = -1\}. \tag{6}$$

Below we use the following non-standard notation to reduce the number of indices.

Notation. For any expression M, such that $M_i = M(x_i, y_i, z_i, \ldots)$, M_i is denoted by $\{M(x, y, z, \ldots)\}_i$.

The problem on which we focus is the maximization of the expected profit from the m projects over the set F, the set of all aggregative decision rules which take into account the order constraint. That is,

[6] We cannot assign 1 or -1 to an individual decision profile of a project, because the collective decision depends upon the other profiles of the other projects. Thus, we need to rely on the combination of all m profiles.

$$\max_{f\in F}\sum_{i=1}^{m}E_i, \tag{7}$$

where E_i is the expected profit from project t_i; i.e.

$$E_i=\{\alpha[B(1{:}1)\Pi(f{:}1)+B(-1{:}1)(1-\Pi(f{:}1))] \\ +(1-\alpha)[B(-1{:}-1)\Pi(f{:}-1)+B(1{:}-1)(1-\Pi(f{:}-1))]\}_i \tag{8}$$

or

$$E_i=\{\alpha B(1)\Pi(f{:}1)+(1-\alpha)B(-1)\Pi(f{:}-1)+(\alpha B(-1{:}1) \\ +(1-\alpha)B(1{:}-1))\}_i. \tag{9}$$

Denote by $g_i(\bar{x}/1)$ and $g_i(\bar{x}/-1)$ the probabilities of obtaining the decision profile $\bar{x}^i$, given $s_i=1$ and $s_i=-1$ for project t_i, respectively. That is,

$$g_i(\bar{x}/1)=\left\{\prod_{j:x_j=1}p_j^1\prod_{j:x_j=-1}(1-p_j^1)\right\}_i, \tag{10}$$

$$g_i(\bar{x}/-1)=\left\{\prod_{j:x_j=-1}p_j^2\prod_{j:x_j=1}(1-p_j^2)\right\}_i. \tag{11}$$

For a given decision rule f,

$$\Pi_i(f{:}1)=\sum_{(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^m)\in X_i(1/f)}g_i(\bar{x}/1)\prod_{j=1,j\neq i}^{m}\{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)\}_j \tag{12}$$

$$\Pi_i(f{:}-1)=\sum_{(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^m)\in X_i(-1/f)}g_i(\bar{x}/-1)\prod_{j=1,j\neq i}^{m} \\ \times\{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)\}_j. \tag{13}$$

In writing $\Pi_i(f{:}1)$, we assume that the state of nature is 1 only for project t_i. We do not know the state of nature for the other projects, so we consider both states of nature. Similarly, for $\Pi_i(f{:}-1)$, the state of nature is assumed to be -1 only for project t_i, while for the other projects both states of nature are considered.

Notice that the solution of the maximization problem on which we focus is also the solution to the following problem:

$$\max_f\sum_{i=1}^{m}\{\alpha B(1)\Pi(f{:}1)+(1-\alpha)B(-1)\Pi(f{:}-1)+C\}_i, \tag{14}$$

where C is a constant, or

$$\max_f \sum_{i=1}^{m} \Bigg[\alpha_i B_i(1) \sum_{(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^m)\in X_i(1/f)} g_i(\bar{x}/1)\cdot \prod_{j=1,j\neq i}^{m} \{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)\}_j$$
$$+(1-\alpha)_i B_i(-1) \sum_{(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^m)\in X_i(-1/f)} g_i(\bar{x}/-1)\cdot \prod_{j=1,j\neq i}^{m} \tag{15}$$
$$\times \{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)\}_j + C_i \Bigg].$$

This can be written in the following way:

$$\max_f \sum_{i=1}^{m} \Bigg[\sum_{(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^m)\in X_i(1/f)} \left\{ \frac{\alpha B(1) g(\bar{x}/1)}{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)} \right\}_i \cdot \prod_{j=1}^{m} \{\alpha g(\bar{x}/1)$$
$$+(1-\alpha)g(\bar{x}/-1)\}_j + \sum_{(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^m)\in X_i(-1/f)} \left\{ \frac{(1-\alpha)B(-1)g(\bar{x}/-1)}{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)} \right\}_i \tag{16}$$
$$\cdot \prod_{j=1}^{m} \{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)\}_j + C_i \Bigg]$$

or

$$\max_f \sum_{j\in(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^m)} \Bigg[\sum_{i\in N'} a_i + \sum_{i\in N-N'} b_i \Bigg] \cdot M_j + \sum_{i=1}^{m} C_i, \tag{17}$$

where

$$a_i = \left\{ \frac{\alpha B(1) g(\bar{x}/1)}{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)} \right\}_i$$
$$b_i = \left\{ \frac{(1-\alpha)B(-1)g(\bar{x}/-1)}{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)} \right\}_i$$

for a combination $j\in(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m)$

$$M_j = \prod_{\ell=1}^{m} \{\alpha g(\bar{x}/1)+(1-\alpha)g(\bar{x}/-1)\}_\ell$$
$$N' = \{t_i \in N/f_i = 1\} \qquad N-N' = \{t_i \in N/f_i = -1\}.$$

Solving the maximization problem on which we focus (17) is done by solving the following problem (18) for any combination of decision profiles. This is due to the fact that the solution of the maximization problem is for any combination of decision profiles; therefore, M_j is not a function of f. Also the C_is are constants

$$\max_f \sum_{i\in N'} a_i + \sum_{i\in N-N'} b_i. \tag{18}$$

The following notation will be useful. Let us assign to a given combination of decision profiles the expected opportunity profit, i.e. the profit from accepting project t_i rather than rejecting it $(a_i - b_i)$, by ΔE_i^*, that is,[7]

$$\Delta E_i^* = \left\{ \frac{\alpha B(1)g(\bar{x}/1) - (1-\alpha)B(-1)g(\bar{x}/-1)}{\alpha g(\bar{x}/1) + (1-\alpha)g(\bar{x}/-1)} \right\}_i. \tag{19}$$

4. Optimization of the collective decision-making process while preserving an order constraint

Without the order constraint, our problem is reduced to the special setting analyzed by earlier literature. The optimal aggregative decision is to accept every project (task) t_i in which $a_i > b_i$, that is $\Delta E_i^* > 0$ (see Ben-Yashar and Nitzan (1997)). But in this paper, this result is extended and we refer to two types of partial order constraints.

4.1. The first type of partial order constraints — 'OR' constraints

We refer first to the first type of partial order constraints. In this case, we are unable to accept all of the projects because of preceding projects (t_j) in which $b_j > a_j$, which therefore may be rejected. The optimal aggregative decision rule f could offer the possibility of accepting every project t_i in which $a_i > b_i$ where all preceding projects t_j should be accepted. However, this possibility does not provide the optimal result. To demonstrate this, assume that t_j is a predecessor of t_i. Also assume that $a_i > b_i$, but $b_j > a_j$. Occasionally, it is worthwhile accepting the preceding project t_j even if $b_j > a_j$, because project t_i can then be accepted. The alternative of rejecting the two projects is worse than the alternative of accepting them both. That is, $a_i + a_j > b_i + b_j$. There are situations in which, although for one project it may not be worthwhile to accept the preceding project t_j, it is possible that for a number of projects, project t_j should be accepted. Assume that t_i and t_ℓ should be accepted; i.e. $a_i > b_i$ and $a_\ell > b_\ell$, but $t_j \preceq t_i$ and $t_j \preceq t_\ell$. Also assume that for the sake of one project only, it is not worth accepting t_j. That is, $a_i + a_j < b_i + b_j$ and $a_\ell + a_j < b_\ell + b_j$. If $a_i + a_\ell + a_j > b_i + b_\ell + b_j$, then the alternative of accepting all of them is preferable over the alternative of rejecting all of them.

Note that, in order to maximize the expected profit, we must examine which alternative is preferable: to reject the project or accept the project as well as its predecessors. This is done by the sign of $\Sigma \Delta E_i^*$ when the sum is taken over the relevant projects. However, there remains a complex issue of which projects are relevant. We now present special examples.

Example 1. Assume that $t_1 \preceq t_2$, $t_3 \preceq t_2$, $t_3 \preceq t_4$, $t_5 \preceq t_4$, t_2 and t_4 have a positive value of ΔE_i^*; t_1, t_3 and t_5 have a negative value of ΔE_i^*. The ΔE_i^* value of each project t_i is

[7] ΔE_i^* can be also interpreted as the difference between the expected net profit when the state of nature is 1 and when it is -1, for a given decision profile.

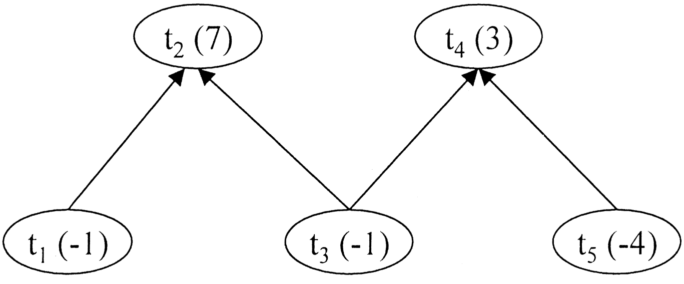

Fig. 1. A graphical representation of the partial order between the projects of Example 1. The optimal alternative of this example is to accept projects t_1, t_2 and t_3.

shown on the nodes in Fig. 1. Even though $\Sigma_{i=1}^{5} \Delta E_i^* > 0$, it is clear that it is better to accept only t_1, t_2, t_3, since $\Sigma_{i=1}^{3} \Delta E_i^* > 0$ and $\Sigma_{i=4}^{5} \Delta E_i^* < 0$.

Example 2. Recall the holiday resorts example presented in the Introduction. In Fig. 2 we describe an investor who considers constructing three holiday resorts t_1, t_3 and t_5. The ΔE_i^* value of each project t_i is denoted on its node. (t_1, t_3 and t_5 have positive values of ΔE_i^*.) Project t_2 consists of the construction of an access road that has to be approved before resorts t_1 and t_3 are approved. Similarly, project t_4 the construction of a second access road must be approved initially for resorts t_3 and t_5, and finally project t_6, the construction of a third access road must be approved before resort t_5 can be approved. Note that the construction of each of these access roads does not pay off (i.e. it has a negative value of ΔE_i^*), but evaluating the projects as a whole justifies the construction of resorts t_1 and t_3 and thus requires, initially, the construction of two access roads, i.e. carrying out t_2 and t_4.

A more complex example appears in Appendix A.

It is easy to see the straightforward exponential solution: for any combination of decision profiles, choose N' from all $\bar{N} \subseteq N$, such that the projects in $\bar{N}$ preserve the order constraints, with the highest value of $\Sigma_{i \in \bar{N}} a_i + \Sigma_{i \in N - \bar{N}} b_i$. Consequently, the aggregation rule f will assign 1 to any project in N' and -1 to all other projects.

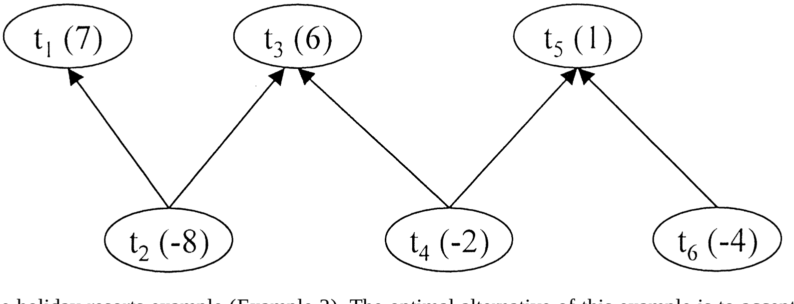

Fig. 2. The holiday resorts example (Example 2). The optimal alternative of this example is to accept projects t_1, $t_{2,}$, t_3 and t_4.

The worst case requires evaluation of 2^m sets of projects and therefore the proposed solution is exponential. An alternative solution which is more efficient is presented in the following theorem.

Theorem 1. *The maximization problem* (18) *under the* '*OR*' *constraints can be solved in polynomial time*.

Proof. A polynomial solution to the problem is based on a reduction of this problem to a known problem of finding a maximum weight closed subset of a directed acyclic graph which has a polynomial solution in the literature (Ahuja et al., 1993).

The maximum weight closed subset problem can be defined as follows: given a directed acyclic graph (DAG), $G=(V, E)$, where V is the set of vertices and E is the set of edges, each vertex ν having positive/negative weight, W_ν, find a closed subset of maximum weight. A *closed* subset is one, in which, if a vertex ν is in the set, then all predecessors of ν must also be in the set (the predecessors of vertex ν are all the vertices that have a directed path to ν).

We can rewrite our problem in the form

$$\max_f \sum_{i\in N'} (a_i - b_i) + \sum_{i\in N} b_i \tag{20}$$

or

$$\max_f \sum_{i\in N'} \Delta E_i^* + \sum_{i\in N} b_i. \tag{21}$$

Notice that the solution of the maximization problem on which we focus is also the solution of the problem:

$$\max_f \sum_{i\in N'} \Delta E_i^*. \tag{22}$$

From this representation one can see that this is the same as finding a closed subset of maximum weight in which each vertex ν represents a project; the edges E represent the partial order, and the weight, W_ν, is represented by ΔE_ν^*.

Given the complexity of finding a maximum weight closed subset (Ahuja et al., 1993), the solution to our problem is now a polynomial of order 3 in the number of projects ($O(m^3)$). □

The main definitions that are needed for finding a maximum weight closed subset and an example are presented in Appendix B.

4.2. The second type of partial order constraints — 'positive OR' constraints

Now we refer to the second type of partial order constraints. In this case, we are unable to reject all of these projects, t_i in which $b_i > a_i$, that is, $\Delta E_i^* < 0$, because of preceding projects to $\neg t_i$ which have been rejected.

We now prove that this problem is NP-complete (i.e. which is, in general, considered to be intractable) by way of reduction from the problem of Vertex Cover.

Theorem 2. *The maximization problem* (18) *under the 'positive OR' constraints is NP-complete*.

Proof. We prove that this problem is NP-complete by way of reduction from the problem of Vertex Cover.

The Vertex Cover problem can be defined as follows: given a graph G, $G=(V, E)$, where V is the set of vertices and E is the set of edges, and given an integer K, 'is there a subset S of vertices, of size $\leq K$, that covers all the edges?' An edge is *covered* if at least one of its endpoints is in S.

Our problem can be simplified by assuming the special case in which each project has a value of -1. We ask the question: 'given an integer K', is there a subset of projects that satisfies this type of partial order constraints with a total profit $\geq K'$?'

One can see that the special case is the same as finding a Vertex Cover in which each vertex v represents a project, and for each edge $e=(u, v)$ in E, we will have a 'positive OR' constraint that says that project u or project v needs to be done.[8] A Vertex Cover of size $\leq K$ is the same set S' of projects with profit $\geq K'$, where $K' = -K$ that satisfied the 'positive OR' constraints. The subset of vertices of size $\leq K$ that covers all the edges, is a subset S' of projects with a total profit $\geq -K$ (multiply both sides by -1).

As seen in Garey and Johnson (1979) the problem of Vertex Cover is NP-complete.

We showed that this special case can be reduced from the Vertex Cover which is NP-complete, therefore, our problem is NP-complete. □

Note that for this type of partial order constraints, in which there are t_i and t_j such that $t_j \preceq \neg t_i$, the reduction described in Theorem 1 to the first type of partial order constraints (in which there are t_i and t_j such that $t_j \preceq t_i$) cannot be performed. The reason is that the reduction to the problem of finding a maximum weight closed subset requires that each project have a corresponding vertex in the directed graph. In order to apply this reduction to the second type of partial order constraints, each project must be considered both as a positive project (t_i) and as a separate negative project ($\neg t_i$). Hence, two vertices will be obtained, one for t_i and the other for $\neg t_i$. Clearly, only one of these projects can be accepted and in this reduction such a constraint cannot be defined since, as mentioned, the graph is directed.

5. Application of the model to information filtering

This 'constrained' dichotomous choice model can be applied to alternative classification decisions, such as information filtering. In recent years, on-line information has become overwhelming and an urgent need has arisen for systems which can help in filtering or searching for relevant information. For example, if a user who is limited in time wishes to search through a large set of documents, the system should automatically recommend documents worth reading. Recommendations can be based on other people's

[8] As we have shown, this type of partial order constraints can be considered 'positive OR' constraints, since $t_j \preceq \neg t_i$ is equivalent to $t_j \vee t_i$, implying that the final decision must approve either t_i or t_j or both.

opinions about these documents and the similarity between their interests and the user's interests (Shardanand and Maes, 1995). Note that the recommendations are based on a group of people and not on one person only because this improves the recommendations concerning information filtering (Condorcet [1785],1994).[9] The problem is then to lay out rules which decide which documents should be recommended, based on peoples' opinions about a particular document and the similarity between these people and the user.

This problem can be regarded as decisions made by a committee. Here the projects are the documents, and accepting a project corresponds to recommending a document. A 'good project' is a document which is of interest to the user and should be recommended to him, while a bad project is one which is of no interest to the user and should not be recommended. Each member of the committee has to decide whether a particular document is of interest and should therefore be recommended. A committee member's decisional skills are based on the similarity between his interests and the user's interests. That is, as the similarity between the user's interests and those of the committee member increases, the decisional skills of the committee member increase. That is, there is a higher probability that a document which is of interest to the committee member will also be of interest to the user. In order to find these decisional skills, a preliminary step should be taken, where both the user and the committee members are given a set of documents and are asked to indicate, for each document, whether it is of interest to them or not. This information can be used to calculate the decisional skills (the probabilities). Note that the decisions and the decisional skills of each member are independent across individuals.

When there is a need to make recommendations with respect to a set of documents with which the user is not familiar, the opinions of the committee members are used. That is, each committee member has an opinion whether a given document in the set is of interest (i.e. assigning '1' to the document) or whether it is uninteresting (i.e. assigning '-1' to the document). Note that, since the number of users is huge, the decisions of each committee member are given, regardless of the user, and hence the collective information can be used for making recommendations for many users. Using the decisional vector of all the committee members with respect to each document, as well as their pre-computed decisional skills, recommendation can be given based on the method for optimal group decision-making.

In some situations, there may be a partial order constraint on the documents. The intuitive meaning of document D1 preceding D2, according to the partial order, is that D1 is a prerequisite of D2; i.e. reading (or viewing) D2 requires information provided in D1. In such situations, the solution for optimal collective decision-making presented in Section 4 can be applied in order to decide which documents should be recommended to the user.

In some situations, there may be more general partial order constraints on the documents. The intuitive meaning of such a D1$\vee$D2 constraint is that D2 and/or D1

[9] The statement that majorities are more likely to select the correct alternative than any single individual is attributed to Condorcet [1785], (1994).

must be read and in no case will neither be read. In such situations the solution is, in general, considered to be intractable.

6. Conclusion

This paper has presented optimal collective dichotomous choices under partial order constraints in the general dichotomous choice framework.

There are a number of issues not dealt with here, but that would be of further interest. For example, until now, the optimal collective decision was a function of the individual decisions in cases where these decisions did not take into consideration the existence of other projects nor the existence of the constraints. It would be interesting to see if and how the optimal collective process and the decision would change as a result of individual decisions taking these factors into account, and if such a change would be profitable.

Another question discusses the advantages of a method in which each individual ranks each project instead of giving a binary decision. In the example of recommending documents, each individual would need to assign a score to each document. On the basis of these rankings, the decision about which documents to recommend will be made.

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Appendix A

Example. Sometimes there are two projects that constitute a prerequisite for a third project, while each also serves as a prerequisite for an additional project. For example, deciding to read books in mathematics and books in game theory serves as a prerequisite for deciding to read articles in economics. Moreover, books in mathematics also serve as prerequisites for physics articles, and books on game theory serve as prerequisites for articles in political science, see Fig. 3.

Appendix B

The definitions described in this appendix are taken from Ahuja et al. (1993), Even (1979) and Ford and Fulkerson (1962).

The max weight closed subset problem can be solved efficiently by reducing it to

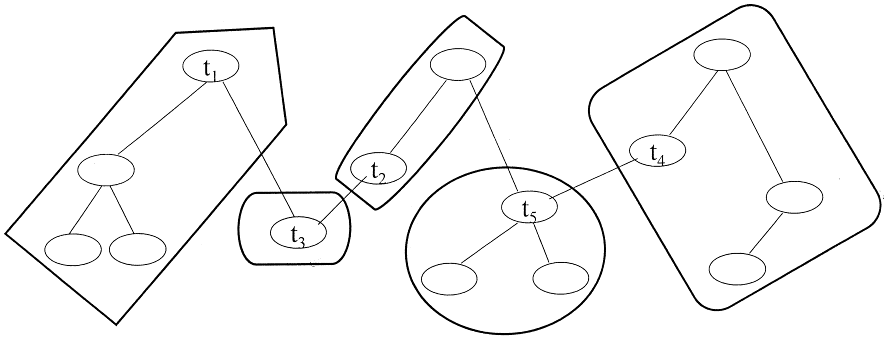

Fig. 3. A graphical representation of the partial order between the projects of the example discussed in Appendix A.

computing the minimum $s - t$ cut in the following directed network, which can be done by a single max flow computation.

A *network* consists of the following data.

1. A finite digraph $G = (V, E)$, with no self-loops and no parallel edges.
2. Two vertices s and t are specified; s is called the *source* and t, the *sink*.
3. Each edge $e \in E$ is assigned a non-negative number $c(e)$, called the *capacity* of e.

The relevant directed network NN is comprised of the following.

1. The vertex set of NN is $V \cup \{s, t\}$, where s and t are two new vertices.
2. For each vertex ν of a negative weight, add the edge (s, ν) with capacity $|W_\nu|$ to NN.
3. For each vertex ν of a positive weight, add the edge (ν, t) with capacity W_ν to NN.
4. All edges of G are added to NN, and each of these edges has infinite capacity.

Let S be a subset of vertices such that $s \in S$ and $t \not\in S$. $\bar{S}$ is the complement of S; i.e. $\bar{S} = V \cup \{s, t\} - S$. Let $(S; \bar{S})$ be the set of edges of G whose start-vertex is in S and end-vertex is in $\bar{S}$. The set $(\bar{S}; S)$ is defined similarly. The set of edges connecting vertices of S with $\bar{S}$ (in both directions) is called the cut defined by S.

Let us denote by $c(S)$ the *capacity of the cut* determined by S, which is defined as follows:

$$c(S) = \sum_{e \in (S; \bar{S})} c(e).$$

A minimum $s - t$ cut is an $s - t$ cut of minimum total capacity.

Consider a minimum $s - t$ cut in NN. Let P be the positive weight vertices whose edges to the sink t are not in the min-cut. It can be shown that the union of the set P and its predecessors is a maximum-weight closed subset of G. For a proof, see Ahuja et al. (1993).

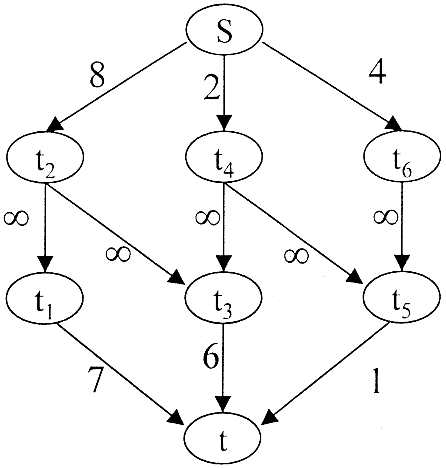

Fig. 4. The relevant network of the graph in Fig. 2.

Note that the minimum $s - t$ cut can be found by a single max flow computation (Even, 1979).

Example. Consider the graph G shown in Fig. 2 and the relevant network in Fig. 4. Next to each edge e, $c(e)$ is written.

$S = \{s, t_6, t_5\}$ defines a minimum cut. Since P is the positive weight vertices whose edges to the sink t are not in the min-cut, then $P = \{t_1, t_3\}$. The union of P and its predecessors is a maximum-weight closed subset, and therefore $N' = \{t_1, t_2, t_3, t_4\}$.

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